Unit 2

Subtraction of Whole Numbers

Throughout their work with subtraction, students should be encouraged to use estimation to check if their answers are reasonable. A visual model to aid students with the concept of rounding (Roller Coaster Rounding) is on pages 143 and 144. Reproducible lessons for estimating differences using front-end estimation and for estimating using compatible numbers (for all operations) are on pages 153 and 176, respectively. (Answers are on pages 195 and 199, respectively.)

Pages 151 through 162 provide instructional games and follow-up activities (reproducible) to support subtraction (and addition) concepts.

Name ____________________ Date ______________ Class ______________

Diagnostic Test
Subtraction of Whole Numbers

Multiple Choice: Circle the correct answer. If your answer is not given, circle *Not here.*

Part 1

1. $14 - 8 = \square$ A 8 B 6 C 22 D 14 E Not here

2. $13 - 5 = \square$ A 8 B 18 C 12 D 7 E Not here

3. $15 - \square = 6$ A 21 B 11 C 8 D 9 E Not here

4. $\square - 8 = 6$ A 2 B 9 C 14 D 16 E Not here

5. Which addition problem can be used to find the value of $\square$ in $16 - 7 = \square$?
A $\square + 7 = 16$ B $7 + 16 = \square$ C $16 + \square = 7$ D $\square - 16 = 7$ E Not here

Part 2

6. $\begin{array}{r} 87 \\ -\ \ 7 \\ \hline \end{array}$ A 10 B 0 C 80 D 8 E Not here

7. $45 - 3$ A 15 B 42 C 12 D 32 E Not here

8. $648 - 40$ A 608 B 248 C 640 D 5,918 E Not here

9. $\begin{array}{r} 158 \\ -\ \ 56 \\ \hline \end{array}$ A 912 B 102 C 1,112 D 12 E Not here

10. $\begin{array}{r} 867 \\ -\ \ 50 \\ \hline \end{array}$ A 617 B 817 C 17 D 810 E Not here

Part 3

11.	$\begin{array}{r} 84 \\ -\ 36 \\ \hline \end{array}$	A 52	B 58	C 28	D 48	E Not here
12.	821 – 63	A 842	B 868	C 758	D 191	E Not here
13.	$\begin{array}{r} 852 \\ -\ 281 \\ \hline \end{array}$	A 670	B 571	C 631	D 5,611	E Not here
14.	$\begin{array}{r} 415 \\ -\ 193 \\ \hline \end{array}$	A 322	B 222	C 321	D 2,112	E Not here
15.	$\begin{array}{r} 9{,}218 \\ -\ 7{,}395 \\ \hline \end{array}$	A 1,823	B 2,923	C 2,183	D 2,912	E Not here

Part 4

16.	$\begin{array}{r} 970 \\ -\ \ \ 6 \\ \hline \end{array}$	A 304	B 964	C 970	D 854	E Not here
17.	$\begin{array}{r} 870 \\ -\ 363 \\ \hline \end{array}$	A 507	B 4,107	C 57	D 513	E Not here
18.	$\begin{array}{r} 705 \\ -\ 289 \\ \hline \end{array}$	A 326	B 584	C 461	D 506	E Not here
19.	$\begin{array}{r} 5{,}004 \\ -\ 610 \\ \hline \end{array}$	A 394	B 5,000	C 5,490	D 3,394	E Not here
20.	$\begin{array}{r} 6{,}000 \\ -\ 1{,}753 \\ \hline \end{array}$	A 4,357	B 5,753	C 4,247	D 5,357	E Not here

ITEM ANALYSIS FOR DIAGNOSTIC TEST

Subtraction of Whole Numbers

Using the Item Analysis Table

- The correct answer for each item on the Diagnostic Test is indicated by a ✓ in the Item Analysis Table on page 54.
- Each incorrect answer choice is keyed to a specific error pattern and corresponding Intervention Activity found on pages 55 through 72. Because each item on the Diagnostic Test is an item that is analyzed in one of the error patterns, teachers may be able to use the Intervention Activities with identical problems that students may have missed on the test.
- Students should be encouraged to circle *Not here* if their obtained answer is not one of the given answer choices. Although *Not here* is never a correct answer on the Diagnostic Test, the use of this answer choice should aid in the diagnostic process. The intention is that students who do not see their obtained answer among the choices will select *Not here* rather than guess at one of the other choices. This should strengthen the likelihood that students who select an incorrect answer choice actually made the error associated with the error pattern.
- The Item Analysis Table should only be used as a *guide.* Although many errors are procedural in nature, others may be due to an incorrect recall of facts or to carelessness. A diagnostic test is just one of many tools that should be considered when assessing student work and making prescriptive decisions. Before being certain that a student has a misconception about a procedure or concept, further analysis may be needed (see below). This is especially true for students who frequently select *Not here* as an answer choice.
- A set of practice exercises, keyed to each of the four parts of the Diagnostic Test, is provided on page 74. Because the four parts of the set of practice exercises match the four parts on the Diagnostic Test, the set of practice exercises could be used as a posttest.

Using Teacher-Directed Questioning and Journaling

Discussions and observations should be used to help distinguish misconceptions about concepts and procedures from student carelessness or lack of fact recall. This should be done in a *positive* manner—with the clear purpose being to "get inside student thinking." The Intervention Activities are replete with teacher-directed questioning, frequently asking students to *explain their reasoning.* Students should also be asked to *write* about their thinking as they work through an algorithm—and, when alternative algorithms are used, explain why they may prefer one algorithm over another. You may also want students to write word problems based on *subtraction*—and then explain why subtraction can be used to solve them. This would be a good time to discuss with students the various actions and problem structures for subtraction (see pages 18–19).

Additional Resources for Subtraction

- A lesson providing a visual model for *rounding* is on pages 143 and 144. A lesson on using *front-end estimation* to check differences for reasonableness is on page 153; a lesson on using *compatible numbers* for estimation (for all operations) is on page 176. These lessons may be used at any point in the instructional process. When students engage in estimation activities, they should discuss why they believe a computed answer may or may not be reasonable.
- Instructional games and follow-up activities for subtraction concepts are on pages 151 through 162. This material may be used at any point in the instructional process. The games *Balance the Number Sentence* and *Target Math* may be used as vehicles for observing student behavior in trial-and-error thinking, computation, and problem solving. The follow-up activity *Editor Error Search* provides a rich opportunity for students to play the role of editor to find and correct errors in a manuscript. Students also are asked to discuss the error patterns they observe in the manuscript.

ITEM ANALYSIS TABLE

The correct answer for each item on the Diagnostic Test is indicated by a ✓ in this table.

	Item	Answer Choices A	Answer Choices B	Answer Choices C	Answer Choices D	Topic	Practice Exercises
	1	Error 1	✓	Error 1	Error 1	Subtraction facts	Part 1, p. 74
	2	✓	Error 1	Error 1	Error 1	Subtraction facts	Part 1, p. 74
	3	Error 1	Error 1	Error 1	✓	Subtraction facts	Part 1, p. 74
	4	Error 1	Error 1	✓	Error 1	Subtraction facts	Part 1, p. 74
Part 1	5	✓	Error 1	Error 1	Error 1	Subtraction facts	Part 1, p. 74
	6	Error 3	Error 6a	✓	Error 11b	Subtraction without renaming	Part 2, p. 74
	7	Error 4	✓	Error 3	Error 6b	Subtraction without renaming	Part 2, p. 74
	8	✓	Error 4	Error 11	Error 9	Subtraction without renaming	Part 2, p. 74
	9	Error 9	✓	Errors 2 & 9	Error 11b	Subtraction without renaming	Part 2, p. 74
Part 2	10	Error 6b	✓	Error 6a	Error 11a	Subtraction without renaming	Part 2, p. 74
	11	Error 2	Error 5	Error 10	✓	Subtraction with renaming	Part 3, p. 74
	12	Error 2	Error 5	✓	Error 4	Subtraction with renaming	Part 3, p. 74
	13	Error 7	✓	Error 2	Error 9	Subtraction with renaming	Part 3, p. 74
	14	Error 5	✓	Error 7	Error 9	Subtraction with renaming	Part 3, p. 74
Part 3	15	✓	Error 5	Error 2	Error 7	Subtraction with renaming	Part 3, p. 74
	16	Error 3	✓	Error 11a	Error 6b	Subtraction with renaming involving 0 or 0s	Part 4, p. 74
	17	✓	Error 10	Error 11b	Error 2	Subtraction with renaming involving 0 or 0s	Part 4, p. 74
	18	Error 8	Error 2	✓	Error 10	Subtraction with renaming involving 0 or 0s	Part 4, p. 74
	19	Error 6a	Error 11a	Error 8	✓	Subtraction with renaming involving 0 or 0s	Part 4, p. 74
Part 4	20	Error 8	Error 2	✓	Errors 5 & 8	Subtraction with renaming involving 0 or 0s	Part 4, p. 74

ERROR PATTERNS & INTERVENTION ACTIVITIES

Subtraction of Whole Numbers

Error Pattern 1

Some students have difficulty recalling basic subtraction facts (through 18 – 9). A lack of understanding of the do/undo (inverse) relationship between addition and subtraction is often a contributing factor to making these errors.

Intervention

Have students "think addition" when they do subtraction by finding the missing part of a subtraction sentence. For example, for 14 – 6 = □, have students think (from right to left), "*What* plus 6 makes 14?" If students cannot recall that 8 + 6 = 14, have them make or use an addition table to help them recall the facts. (A blank table and a completed table are provided on pages 188–189.)

Students could find 6 in the shaded column on the left in the table, then slide across (along the dashed arrow) to 14, and finally slide up to see that 8 (circled in the top row) answers, "*What* plus 6 makes 14?" Repeat the above for 14 – 8 = □. The solid arrows in the table show that 6 answers the question, "*What* plus 8 makes 14?"

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Repeat with subtraction sentences where the missing part is *not* on the right-hand side of the equal sign, such as 14 – □ = 8 or □ – 8 = 6. Such sentences encourage students to think of the equal sign as meaning equality and balance (rather than as "find the answer").

As you relate subtraction facts to the addition facts students already know, bring out the idea of fact families. Provide an addition sentence, and then have students write the three corresponding related sentences. For 9 + 7 = 16, students should write all four facts in the family as shown below.

9 + 7 = 16	7 + 9 = 16
16 – 7 = 9	16 – 9 = 7

Instructional Game: Balance the +/– Number Sentence!
(See pages 154–157 for game instructions and game pieces. Students work in groups of 2 or 3.)

This "domino-type" game promotes memorization of the facts while having students use trial-and-error thinking to balance addition/subtraction number sentences. Students use playing cards as "dominoes" to match either a number sentence with its solution or a solution with its number sentence. (Shown below, the card with the white 14 was placed next to the card with □ – 8 = 6 because 14 makes that sentence true.)

17 – □ = 9	14	**□ – 8 = 6**	7

Most of the game cards involve the use of the basic facts. However, advise students that those game cards that are based on two-digit addition or subtraction can be solved using *mathematical reasoning* rather than by performing actual computations. For example, to solve 24 + 25 = □ + 24, students can use the *Commutative Property of Addition* to determine that □ = 25. For 31 – 24 = 31 – □, students should think about the value for □ that puts both sides in balance. The value for □, of course, is 24.

Error Pattern 2

Some students subtract the lesser digit from the greater digit in each place-value position, ignoring order (and renaming).

$$\begin{array}{r} 84 \\ -\,36 \\ \hline 52 \end{array} \qquad \begin{array}{r} 821 \\ -\;\;63 \\ \hline 842 \end{array} \qquad \begin{array}{r} 870 \\ -\;\;76 \\ \hline 806 \end{array} \qquad \begin{array}{r} 9{,}218 \\ -\,7{,}395 \\ \hline 2{,}183 \end{array}$$

Intervention

Note: Base-ten blocks may be used instead of play money in the Interventions for Error Patterns 2 through 6.

Supply a subtraction exercise on a place-value grid, a place-value mat on tagboard, and play money. (See pages 184–186 for play money, mat, and grids.)

To subtract 36 from 84, guide students to model the top number (the *minuend*) by placing 8 ten-dollar bills and 4 one-dollar bills on the mat to represent 84. Point out that the *subtrahend* tells us how much we are to *take away* in each place-value position. In terms of vocabulary, to help students connect "subtrahend" to "the bottom number of a subtraction exercise," have them think of *subway.*

T	O
8	4
– 3	6

Tens		Ones	
10	10	1	1
10	10	1	1
10	10		
10	10		

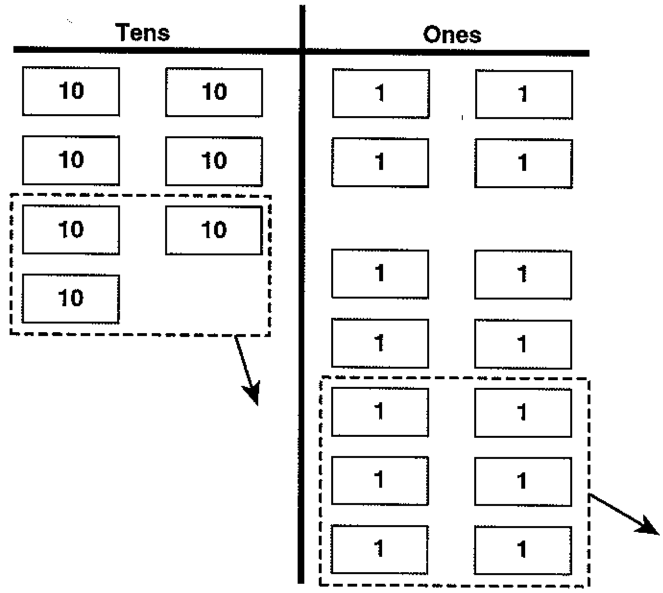
Emphasize that we start with the ones. **Ask,** "Are there enough one-dollar bills on the mat so that you can remove 6 one-dollar bills?" (No.) Note that when you ask whether there are enough one-dollar bills on the mat to remove 6 one-dollar bills, try to avoid language such as "Can you subtract 6 from 4?" To suggest that you *cannot subtract 6 from 4* could lead to later confusion when students study integers and learn that $4 - 6 = -2$.

Ask, "What can we do with the play money we have to get more one-dollar bills on the mat?" (Sample: Trade 1 ten-dollar bill for 10 one-dollar bills.) Guide students in trading 1 ten-dollar bill for 10 one-dollar bills, and record the renaming in the subtraction exercise. (See page 58.)

10 → 1 1 1 1 1 1 1 1 1 1

Ask, "Did this renaming change the total amount of money you have with your ten- and one-dollar bills? Explain." (No. Sample: Before the renaming, we had 8 ten-dollar bills and 4 one-dollar bills, or $80 + $4 = $84. After the renaming, we have 7 ten-dollar bills and 14 one-dollar bills, or $70 + $14 = $84.) Then, have students perform the subtraction in the ones position on the mat by removing 6 one-dollar bills. Instruct students to record the result in the exercise. Then, have them remove 3 ten-dollar bills and record the final result.

T	O
7	14
~~8~~	~~4~~
− 3	6
4	8



Alternative Intervention: Using Mental Math on an Empty Number Line

Some students may gain fluency with subtraction by doing the computation mentally on an empty (open) number line. By using an empty number line, students should (1) realize that answers such as those described in Error Pattern 2 are not reasonable and (2) obtain correct results without using an algorithm (that may have caused them difficulty).

Merits of using an empty number line are detailed in the unit on addition, on page 42. These benefits include the ability to see partial results, gaining practice with compensation (adding or subtracting too much and then compensating), and more. For subtraction, the empty number line also promotes facility with counting by tens (on and off the decade), with bridging tens, and with splitting numbers as described below.

- Students gain facility with counting-down mental computations:

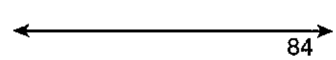
  On the decade: 80 – 10 = 70; 70 – 10 = 60
  Off the decade: 54 – 10 = 44; 44 – 10 = 34

- Students gain facility with splitting the tens and ones—and addressing these separately:

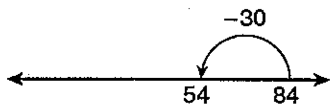
  To find 76 – 9, they may first find 76 – 6 = 70. Then they find 70 – 3, obtaining 67.

Use an empty number line to find 84 – 36:

To find 84 – 36, begin by drawing an empty number line with 84 shown at the far right below the line.

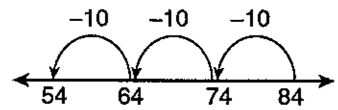


Next, you can subtract 30 by counting down 30 units (to the left) all at once (to obtain 54). Or, you can count down 10 at a time from 84 to 54.



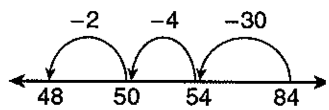
Think: 84 − 30 = 54

or



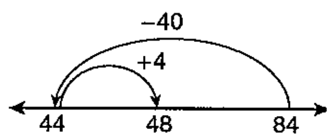
Think: 84 − 10 = 74; 74 − 10 = 64;
64 − 10 = 54

To subtract the remaining 6 from 54, you can first subtract 4 to obtain 50. Then subtract 2 to obtain 48.



Think: 54 − 4 = 50; 50 − 2 = 48.

Another way to subtract 36 from 84 is to use the number line to subtract 40 from 84, obtaining 44. Because you subtracted 4 too much, you compensate by adding 4 to the 44, obtaining 48.



Think: 84 − 40 = 44; 44 + 4 = 48.

Using Student Journaling: Because the open number line permits students to find differences in an individualized way—with students essentially making their own judgments to determine what steps to use—this method lends itself nicely to student journaling. Ask students to write about the steps they use. Also, have them compare and contrast this algorithm with the traditional algorithm.

According to Carroll and Porter (1998), students' writing about their algorithms encourages them "to think through the steps of their algorithms and clarify and refine their ideas about the procedures used. . . . Written descriptions of the algorithm can help to clarify for the teacher whether the student merely made a careless slip or has a deeper misunderstanding of the procedure—something that may not be clear from the use of the algorithm alone" (p. 113).

Error Pattern 3

When subtracting a one-digit number, some students subtract the one-digit number from each digit of the minuend.

$$\begin{array}{rr} 8 & 7 \\ - & 7 \\ \hline 1 & 0 \end{array} \qquad \begin{array}{rr} {}^{7}\cancel{8} & {}^{12}\cancel{2} \\ - & 6 \\ \hline 1 & 6 \end{array} \qquad \begin{array}{rrr} 9 & {}^{6}\cancel{7} & {}^{10}\cancel{0} \\ - & & 6 \\ \hline 3 & 0 & 4 \end{array}$$

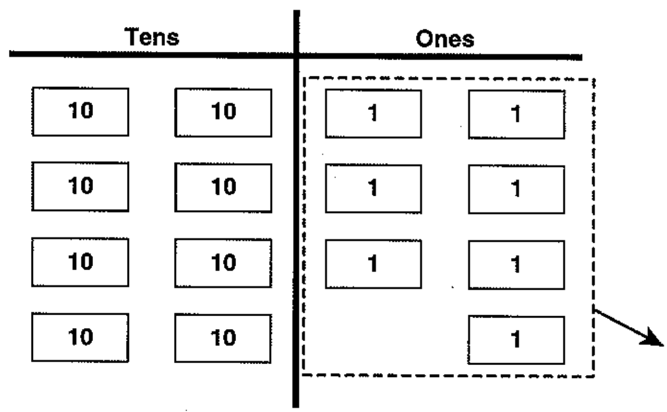
Intervention

Students should be encouraged to use estimation to check if their answer is reasonable. **Ask,** "Can 7 from 87 possibly be just 10?" (No.) "Can 6 from 970 possibly be 304?" (No.)

Supply a subtraction exercise on a place-value grid, a place-value mat on tagboard, and play money (or base-ten blocks). To subtract 7 from 87, guide students in placing 8 ten-dollar bills and 7 one-dollar bills on the mat. Instruct students to remove 7 one-dollar bills and record the result (0) in the subtraction exercise.

Ask, "Are we being asked to remove any tens?" (No.) Emphasize that because there are no tens to remove, you subtract 0 tens from 8 tens. Bring out that you subtract ones from ones, tens from tens, and so on.

$$\begin{array}{r|l} \text{T} & \text{O} \\ 8 & 7 \\ - \quad & 7 \\ \hline 8 & 0 \end{array}$$



Error Pattern 4

When a subtraction problem is posed in horizontal form and there are more digits in the minuend than in the subtrahend, some students fail to align the digits correctly.

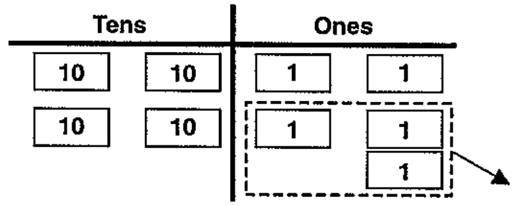
$$45 - 3 \quad \rightarrow \quad \begin{array}{rr} 4 & 5 \\ -3 & \\ \hline 1 & 5 \end{array} \qquad\qquad 648 - 40 \quad \rightarrow \quad \begin{array}{rrr} 6 & 4 & 8 \\ -4 & 0 & \\ \hline 2 & 4 & 8 \end{array}$$

Intervention

Supply grid paper, a place-value mat on tagboard, and play money. For 45 – 3, have students place 4 ten-dollar bills and 5 one-dollar bills on the mat. **Ask,** "Are we subtracting ones or tens?" (Ones.) Guide students in recording the exercise on the grid paper, aligning the ones with the ones, and so on. Have students remove 3 one-dollar bills from the mat and record the subtraction on the grid paper. Emphasize that we subtract ones from ones, and since there are no tens to subtract, we have 4 – 0, or 4 tens, in the answer.

45 – 3 →

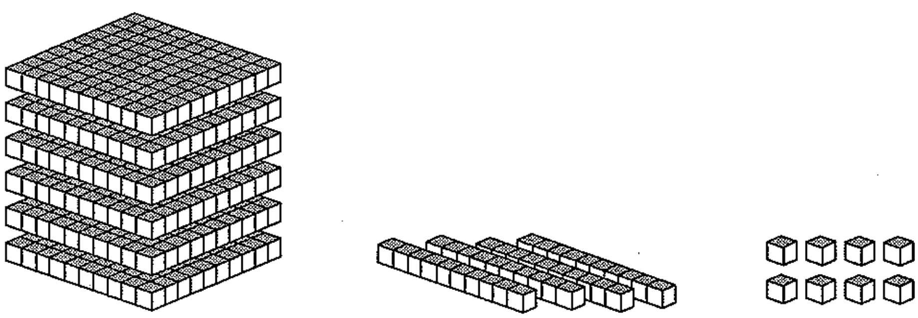
	T	O
	4	5
–		3
	4	2



Alternative Intervention: Using Base-Ten Blocks

Supply students with grid paper and some base-ten blocks. Explain that each flat represents 100, each rod represents 10, and each unit represents 1. Advise students that the blocks will help them maintain the correct place-value positions for the digits as they convert an exercise set up in horizontal form to one set up in vertical form.

To find 648 – 40, have students begin by displaying blocks to represent 648. **Ask,** "How many units are there?" (8 units.) "How many rods are there?" (4 rods.) "How many flats are there?" (6 flats.) Have students write 648 in the minuend on the grid paper as shown below.



6 hundreds 4 tens 8 ones

Then, ask students to represent 40 with the blocks. **Ask,** "How many units are there? (0 units.) **Ask,** "Will we be writing a digit in the ones place of the subtrahend? If so, what digit?" (Yes; 0.) **Ask,** "How many rods are there?" (4 rods.) **Ask,** "Will we be writing a digit in the tens place of the subtrahend? If so, what digit?" (Yes; 4.) **Ask,** "How many flats are there?" (0 flats.) **Ask,** "Will we be writing a digit in the hundreds place of the subtrahend? If so, what digit?" (No.) Have them write 40 in the subtrahend on the grid paper.

648 – 40 →

	H	T	O
	6	4	8
–		4	0

To complete the subtraction, refer students to the base-ten blocks representing the minuend. **Ask,** "When 0 ones are subtracted from 8 ones, how many ones are left?" (8 ones.) **Ask,** "When 4 tens are subtracted from 4 tens, how many tens are left?" (0 tens.) **Ask,** "Will we be subtracting any hundreds from the 6 hundreds that we have?" (No.) Have students complete the subtraction to determine that the difference is 608.

Error Pattern 5

When a subtraction involves renaming, some students do not record all of the renaming process. This generally involves "incomplete trades."

$$\begin{array}{cccc} & & 11 & \\ & 4 & \cancel{1} & 5 \\ - & 1 & 9 & 3 \\ \hline & 3 & 2 & 2 \end{array}$$

The student adds 10 tens to the tens position, but no renaming is done to the hundreds.

Step 1

$$\begin{array}{cccc} & & & 10 \\ & 6 & 4 & \cancel{0} \\ - & 3 & 5 & 9 \\ \hline & & & 1 \end{array}$$

Renaming occurs in the ones position, but not in the tens position

Step 2

$$\begin{array}{cccc} & & 14 & 10 \\ & 6 & \cancel{4} & \cancel{0} \\ - & 3 & 5 & 9 \\ \hline & 3 & 9 & 1 \end{array}$$

Then renaming occurs in the tens position only.

Intervention

Estimation: Determining If an Answer Is Reasonable

All students, especially those who obtain unreasonable answers, should be encouraged to use estimation either before or after computing. When students round 415 and 193 to the nearest hundred, they should see that the answer to 415 – 193 is about 400 – 200, or about 200 (and that, say, 322 is far from the exact answer). Some students, however, find rounding to be a difficult process. The lesson "Roller Coaster Rounding" (pages 143–144) provides a model for students to use to visualize the rounding process.

Some students, especially those who find the rounding process to be difficult, may benefit from using front-end estimation as a way to determine if an answer is reasonable. The lesson "Using Front-End Estimation to Check for Reasonableness: Subtraction" (page 153) teaches students how to use this strategy. A lesson on using *compatible numbers* to make estimates (for all operations) is on page 176.

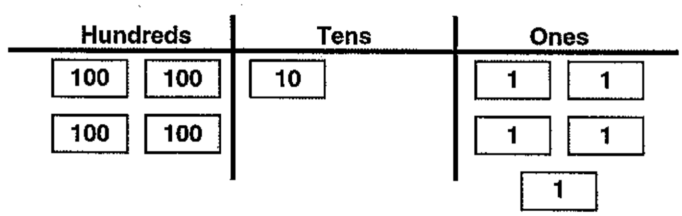
Intervention: Using Play Money or Base-Ten Blocks

Three-digit subtraction with renaming is especially difficult for some students. As Thanheiser (2009) points out, "to explain regrouping from the hundreds to the tens in the (standard) algorithm, a student needs to see the hundred in terms of 1 hundred to explain why we are taking away 1, in terms of 10 tens to explain why we add 10 in the tens place, and in terms of 100 ones to explain the value of the regrouped digit in the tens place" (p. 262).

Activities with students should thus engage them in understanding (1) why renaming is necessary and (2) why the renaming does not change the value of the number.

Supply a subtraction exercise on a place-value grid, a place-value mat on tagboard, and play money (or base-ten blocks). To subtract 193 from 415, guide students in placing 4 hundred-dollar bills, 1 ten-dollar bill, and 5 one-dollar bills on the mat to represent 415.

H	T	O
4	1	5
– 1	9	3

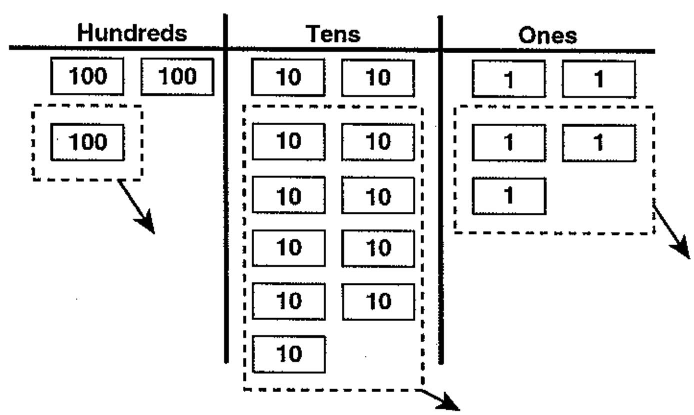


Emphasize that we start with the ones. **Ask,** "Can you remove 3 one-dollar bills?" (Yes.) Have students remove 3 ones and record the result in the subtraction exercise. **Ask,** "Can you remove 9 tens from the mat?" (No.) **Ask,** "What can we do with the play money so that we have more ten-dollar bills?" (Trade 1 hundred-dollar bill for 10 ten-dollar bills.)

$\boxed{100} \rightarrow \boxed{10}\ \boxed{10}\ \boxed{10}\ \boxed{10}\ \boxed{10}\ \boxed{10}\ \boxed{10}\ \boxed{10}\ \boxed{10}\ \boxed{10}$

Have students show this renaming by removing 1 hundred-dollar bill from the hundreds column and replacing it with 10 ten-dollar bills in the tens column. **Ask,** "Did this renaming change the total amount of money you have with your hundred- and ten-dollar bills? Explain." (No. Sample: Before the renaming, we had 4 hundred-dollar bills and 1 ten-dollar bill, or \$400 + \$10 = \$410. After the renaming, we have 3 hundred-dollar bills and 11 ten-dollar bills, or \$300 + \$110 = \$410.) Instruct students to show this renaming in the exercise as shown below. Now have students remove 9 ten-dollar bills and then 1 hundred-dollar bill. Then have students record the results in the exercise.

H	T	O
3	11	
~~4~~	~~1~~	5
– 1	9	3
2	2	2



Error Pattern 6

Error Pattern 6a: When there are more digits in the minuend than in the subtrahend, some students fail to subtract in the positions where the "extra" digits occur (illustrated in the first two examples below).

Error Pattern 6b: When there are more digits in the minuend than in the subtrahend, some students may correctly subtract in the ones position, but "subtract" 1 from each of the "extra" digits (illustrated in the final two examples below).

$$\begin{array}{rr} 9 & 7 \\ - & 4 \\ \hline & 3 \end{array}$$

The student subtracts 4 from 7 but fails to record anything in the tens position.

$$\begin{array}{rr} 7 & 12 \\ \cancel{8} & \cancel{2} \\ - & 6 \\ \hline & 6 \end{array}$$

The student fails to subtract in the tens position.

$$\begin{array}{rr} 6 & 8 \\ - & 7 \\ \hline 5 & 1 \end{array}$$

The student subtracts 7 from 8 but subtracts 1 from the tens digit (rather than 0).

$$\begin{array}{rrr} & 6 & 10 \\ 9 & \cancel{7} & \cancel{0} \\ - & & 6 \\ \hline 8 & 5 & 4 \end{array}$$

The student correctly subtracts in the ones position but subtracts 1 in each of the other positions.

Intervention

Ask students to estimate the differences before they compute. **Ask,** "Does an answer of 3 make sense when you take 4 from 97? Explain." (No. Sample: There are no tens to subtract off. The answer should be close to 90.) Use the Intervention for Error Pattern 3 to emphasize that when no bills are to be removed from a place-value position, you are subtracting 0 from a number.

Error Pattern 7

Some students subtract from left to right. (This error only becomes apparent in computations that involve renaming.)

$$\begin{array}{rrr} & 15 & 1 \\ 8 & \cancel{5} & \cancel{2} \\ -2 & 8 & 1 \\ \hline 6 & 7 & 0 \end{array} \qquad\qquad \begin{array}{rrrr} & & 9 & \\ & 14 & \cancel{10} & 1 \\ 7, & \cancel{4} & \cancel{0} & \cancel{2} \\ -5, & 6 & 8 & 0 \\ \hline 2, & 8 & 1 & 1 \end{array}$$

Intervention

Supply a subtraction exercise on a place-value grid with an index card, revealing only the digits in the ones place. Subtract in the ones position.

H	T	O
8	5	2
−2	8	1

O
2
1
1

Then instruct students to move the card one column to the left, revealing the tens digits. Because renaming is necessary before subtracting the tens, instruct students to move the card one more column to the left to reveal the digits in the hundreds place. Have them record the renaming and subtract the tens and then the hundreds.

T	O
5	2
8	1
	1

	H	T	O
	7	15	
	$\cancel{8}$	$\cancel{5}$	2
−	2	8	1
	5	7	1

Error Pattern 8

Some students rename incorrectly when they have to "cross 0s" in the minuend.

$$\begin{array}{rrrrr} & 5 & 10 & 10 & 10 \\ & \cancel{6}, & \cancel{0} & \cancel{0} & \cancel{0} \\ - & 1, & 7 & 5 & 3 \\ \hline & 4, & 3 & 5 & 7 \end{array}$$

The 1,000 that is "borrowed" from the 6 thousands is spread across three place-value positions as 10 hundreds, 10 tens, and 10 ones.

Step 1

$$\begin{array}{rrrr} & 6 & & 15 \\ & \cancel{7} & 0 & \cancel{5} \\ - & 2 & 8 & 9 \\ \hline & & & 6 \end{array}$$

Renaming is done directly from hundreds to ones. The student renames 7 hundreds as 6 hundreds and 10 ones.

Step 2

$$\begin{array}{rrrr} & 5 & & \\ & \cancel{6} & 10 & 15 \\ & \cancel{7} & \cancel{0} & \cancel{5} \\ - & 2 & 8 & 9 \\ \hline & 3 & 2 & 6 \end{array}$$

Renaming from the hundreds is then done a second time—this time to the tens.

Intervention

Many students find subtracting "across 0s" to be one of the most difficult tasks they experience at this point in their mathematics careers. Some students never master this skill using the traditional subtraction algorithm. It should be noted that the standard algorithm is based on shortcut notation where *individual digits* stand for tens, hundreds, thousands, and so on. When it is necessary to rename multiple times in an exercise, the shortcut can be quite confusing to some students.

The alternative subtraction algorithm described on page 66 and practiced on page 67 utilizes expanded notation to help students visualize the actions that occur in each step. You may want to review expanded notation with students before presenting this algorithm, reminding them that expanded notation is used to rename a number as a sum of the values of the digits in the number. **Ask,** "How do you write 289 in expanded notation?" (200 + 80 + 9.) **Ask,** "How do you write 705 in expanded notation?" (700 + 0 + 5.)

Another alternative subtraction algorithm, *using compensation,* is included with Error Pattern 10—and should benefit students who have difficulty with the renaming process.

Alternative Subtraction Algorithm: Using Expanded Notation

Write the minuend and subtrahend in expanded notation as shown. Parentheses are used to show that we are subtracting the entire number (200 + 80 + 9) from the minuend.

$$\begin{array}{rcrrcrcrl}
705 & \rightarrow & (& 700 & + & 0 & + & 5 &) \\
-\ 289 & \rightarrow & -(& 200 & + & 80 & + & 9 &) \\ \cline{3-9}
\end{array}$$

Subtract in each place-value position beginning with the ones. **Ask,** "Are there enough ones to subtract 9 ones from 5 ones?" (No.) Explain that to get more ones, you can first rename 700 as 600 + 100. Record the 600 in the hundreds position and the 100 in the tens position as shown (crossing out the 700 and 0).

$$\begin{array}{rcrrcrcrl}
 & & & 600 & & 100 & & & \\
705 & \rightarrow & (& \cancel{700} & + & \cancel{0} & + & 5 &) \\
-\ 289 & \rightarrow & -(& 200 & + & 80 & + & 9 &) \\ \cline{3-9}
\end{array}$$

Now, rename 100 + 5 (from the tens and ones positions) as 90 + 15. (This is akin to renaming 10 tens and 5 ones as 9 tens and 15 ones.) Record the 90 in the tens position and the 15 in the ones position as shown (crossing out the 100 and the 5).

$$\begin{array}{rcrrcrcrl}
 & & & & & 90 & & & \\
 & & & 600 & & \cancel{100} & & 15 & \\
705 & \rightarrow & (& \cancel{700} & + & \cancel{0} & + & \cancel{5} &) \\
-\ 289 & \rightarrow & -(& 200 & + & 80 & + & 9 &) \\ \cline{3-9}
\end{array}$$

Now, subtract in each position (15 – 9 = 6; 90 – 80 = 10; 600 – 200 = 400). Then, write the answer in *standard notation:* 400 + 10 + 6 = 416.

$$\begin{array}{rcrrcrcrlcr}
 & & & & & 90 & & & & & \\
 & & & 600 & & \cancel{100} & & 15 & & & \\
705 & \rightarrow & (& \cancel{700} & + & \cancel{0} & + & \cancel{5} &) & & \\
-\ 289 & \rightarrow & -(& 200 & + & 80 & + & 9 &) & & \\ \cline{3-9}
 & & & 400 & + & 10 & + & 6 & & = & 416
\end{array}$$

You may want to photocopy the Guided Problems on page 67 so that students may practice using this alternative algorithm.

Guided Problem 1

Use the alternative algorithm to subtract 485 from 903. First, write each number in expanded notation.

$$\begin{array}{rcl} 903 & \rightarrow & (\ \underline{\quad\quad} + \underline{\quad\quad} + \underline{\quad}\ ) \\ \underline{-485} & \rightarrow & \underline{-(\ \underline{\quad\quad} + \underline{\quad\quad} + \underline{\quad}\ )} \end{array}$$

Since we do not have enough ones to take 5 ones from 3 ones, we rename 900 as 800 + 100. Record the renaming (800 in the hundreds column; 100 in the tens column).

$$\begin{array}{rcl} 903 & \rightarrow & (\ \cancel{900} + \cancel{0} + 3\ ) \\ \underline{-485} & \rightarrow & \underline{-(\ 400 + 80 + 5\ )} \end{array}$$

Now rename 100 + 3 as 90 + 13 (in the tens and ones positions).

$$\begin{array}{rcl} 903 & \rightarrow & (\ \overset{800}{\cancel{900}} + \overset{100}{\cancel{0}} + 3\ ) \\ \underline{-485} & \rightarrow & \underline{-(\ 400 + 80 + 5\ )} \end{array}$$

Now subtract in each place-value position. Then write the difference in standard form.

$$\begin{array}{rcl} 903 & \rightarrow & (\ \overset{800}{\cancel{900}} + \overset{\overset{90}{\cancel{100}}}{\cancel{0}} + \overset{13}{\cancel{3}}\ ) \\ \underline{-485} & \rightarrow & \underline{-(\ 400 + 80 + 5\ )} \end{array}$$

$$\underline{\quad\quad} + \underline{\quad\quad} + \underline{\quad} = \underline{\quad\quad\quad\quad}$$

Guided Problem 2

Use the alternative algorithm to subtract 1,753 from 6,000.

$$\begin{array}{rcl} 6{,}000 & \rightarrow & (\ \overset{5{,}000}{\cancel{6{,}000}} + \overset{1{,}000}{\cancel{0}} + 0 + 0\ ) \\ \underline{-1{,}753} & \rightarrow & \underline{-(\ \underline{\quad\quad} + \underline{\quad\quad} + \underline{\quad\quad} + \underline{\quad}\ )} \end{array}$$

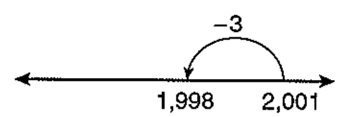
$$\underline{\quad\quad} + \underline{\quad\quad} + \underline{\quad\quad} + \underline{\quad} = \underline{\quad\quad\quad\quad}$$

Alternative Intervention: Using Mental Math on an Empty Number Line

Some students will experience success finding differences when the minuend contains 0s by using mental math on an empty number line. (See the Alternative Intervention for Error Pattern 2.) Consider the exercise 2,001 – 3. According to Reeves and Reeves (2003), to find 2,001 – 3, most students will write the 3 below the 2,001, then subtract by renaming across the 0s, eventually obtaining 1,998 (hopefully). Reeves and Reeves found that more than 95% of the students in their study responded in such a way—even though in subsequent discussions they could readily identify easier ways to find the answer.

Use an empty number line to find 2,001 – 3:

To find 2,001 – 3, write 2,001 at the far right below the number line. To subtract 3, simply count down 3 units to obtain 1,998.



Error Pattern 9

Some students rename when it is not necessary to rename.

$$\begin{array}{rr} 3 & \\ \cancel{4} & 8 \\ -1 & 7 \\ \hline 2 & 1 \end{array}$$

The student erroneously renames 4 tens as 3 tens.

$$\begin{array}{rrr} & 14 & \\ & \cancel{4} & 18 \\ \cancel{1} & \cancel{5} & \cancel{8} \\ - & 5 & 6 \\ \hline & 9 & 12 \end{array}$$

The student unnecessarily renames 5 tens and 8 ones as 4 tens and 18 ones. Then, 6 is subtracted from 18, with 12 recorded in the ones position. The student then correctly renames the hundreds and tens.

$$\begin{array}{rrr} & 16 & \\ 8 & \cancel{6} & 13 \\ \cancel{9} & \cancel{7} & \cancel{3} \\ -4 & 0 & 6 \\ \hline 4 & 16 & 7 \end{array}$$

The student renames 7 tens and 3 ones as 6 tens and 13 ones. Then, 9 hundreds and 6 tens are renamed as 8 hundreds and 16 tens.

$$\begin{array}{rrr} & 18 & \\ 4 & \cancel{8} & 11 \\ \cancel{5} & \cancel{9} & \cancel{0} \\ -4 & 2 & 2 \\ \hline & 16 & 9 \end{array}$$

9 tens and 0 ones are renamed as 8 tens and 11 ones. Then, 5 hundreds and 8 tens are renamed as 4 hundreds and 18 tens.

Intervention

For 158 – 56, display 1 hundred-dollar bill, 5 ten-dollar bills, and 8 one-dollar bills. **Ask,** "Can you take 6 one-dollar bills from 8 one-dollar bills *without* trading?" (Yes.) **Ask,** "Can you take 5 ten-dollar bills from 5 ten-dollar bills *without* trading?" (Yes.) **Ask,** "Can you take 0 hundred-dollar bills from 1 hundred-dollar bill *without* trading?" (Yes.)

Then use the Intervention for Error Pattern 5 to find 590 – 522. **Ask,** "Can you take 2 one-dollar bills from 0 one-dollar bills *without* trading? (No.) **Ask,** "In subtraction, when do you need to trade? When should you *not* trade?" Students should conclude that in a given place-value position, if the digit in the subtrahend *is less than or equal to* the digit in the minuend, no renaming is done. If the digit in the subtrahend *is greater than* the digit in the minuend, then renaming is needed. Have students use the intervention to complete the subtraction.

Error Pattern 10

When renaming is necessary in the ones position, some students rename the digit in the tens position by subtracting the smaller digit in the tens positions from the larger digit (rather than renaming the digit in the minuend as one less ten).

$$\begin{array}{rrr} & 5 & 13 \\ 6 & \cancel{9} & \cancel{3} \\ - & 4 & 5 \\ \hline 6 & 1 & 8 \end{array}$$

Rather than renaming 9 tens and 3 ones as 8 tens and 13 ones, the student renames them as 5 tens and 13 ones. The student obtains the 5 tens by subtracting 4 from 9.

$$\begin{array}{rrrr} & & 16 & \\ & 7 & \cancel{6} & \cancel{12} \\ 9, & \cancel{8} & \cancel{7} & \cancel{2} \\ - & 1 & 7 & 9 \\ \hline 9, & 6 & 9 & 3 \end{array}$$

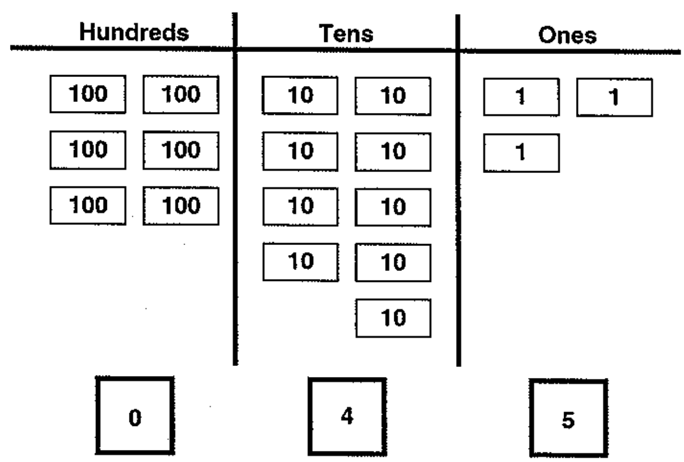
To obtain the renamed tens digit in the minuend, the student subtracts 1 from 7 to obtain 6 tens. Then renaming occurs again, and the student renames 8 hundreds and 6 tens as (8 – 1) or 7 hundreds and 6 tens.

Intervention

Supply a subtraction exercise on a place-value grid and also a place-value mat on tagboard. Also supply each student with a set of digit cards (provided on page 171).

To subtract 45 from 693, guide students in placing 6 hundred-dollar bills, 9 ten-dollar bills, and 3 one-dollar bills on the mat to represent 693. To emphasize the digit to be subtracted in each place-value position, have students place a digit card in each place-value position for each digit of the subtrahend as shown below. So, a 0 card is placed below in the hundreds position (because only a two-digit number is being subtracted), a 4 card is placed below in the tens position, and a 5 card is placed below in the ones position.

H	T	O
6	9	3
–	4	5



Complete the subtraction by using the Intervention for Error Pattern 5. Emphasize that to obtain enough ones to subtract 5 ones, you rename 9 tens and 3 ones as 8 tens and 13 ones. Explain that the 4 tens that are to be subtracted (in the subtrahend) are *not* part of this renaming process. **Ask,** "After you rename, how many tens are left in the minuend? (8 tens.) "How many tens will you have left after you subtract the 4 tens?" (4 tens.)

Alternative Subtraction Algorithm: Using Compensation (Also Known as the "Equal Additions Method" and the "European-Latino Method").

This method is used in most European and Latin countries, among others. According to Ron (1998), "because parents will always try to help their children with mathematics as they know it, it is helpful for teachers to understand the European-Latino algorithm. . . . The phrase 'my family taught me this way' is often heard in the classroom as an explanation for a procedure that students realize was not taught in class" (pp. 115, 117).

Ron suggests that teachers accept such valid alternative algorithms students bring from home rather than disallowing their use. It should be noted, however, that children exposed to both the U.S. traditional and European-Latino algorithms often make errors by mixing the two algorithms. So, asking students to explain their subtraction work is especially important to help teachers sort through any such errors.

By using the concept of *compensation,* this algorithm avoids the "borrowing" process. Thus, students who struggle with the "borrowing" process may especially benefit from using this algorithm. Before presenting the algorithm, develop the concept of compensation by displaying Exercises A, B, C, and D below and then asking the questions that follow.

A	B	C	D
$\begin{array}{r} 57 \\ -34 \\ \hline \end{array}$	$\begin{array}{r} 67 \\ -44 \\ \hline \end{array}$	$\begin{array}{r} 790 \\ -430 \\ \hline \end{array}$	$\begin{array}{r} 890 \\ -530 \\ \hline \end{array}$

Questions:

Find each subtraction result in A and B. What do you notice about each difference? (Each difference is 23.)

What can you say about the minuends and subtrahends in A and B? (Sample: The minuend and subtrahend in B are each 10 more than the minuend and subtrahend in A.)

Find each subtraction result in C and D. What do you notice about each difference? (Each difference is 360.)

What can you say about the minuends and subtrahends in C and D? (Sample: The minuend and subtrahend in D are each 100 more than the minuend and subtrahend in C.)

Resources for Subtraction

(Estimation, Instructional Games, and Follow-Up Activities)

This lesson provides a practical way for students to determine if a subtraction result is reasonable. The process involves subtracting the front-end digits (the most important digits in determining an approximate answer). The process also involves making an adjustment by taking the remaining digits into account. For some students, the adjustment process may be difficult—and for those students it may be omitted. Answers are provided on page 195. (For exercises 16–21, you may want students to also explain their reasoning for determining their Yes/No answers.) *The lesson may be used at any time during the instructional process.*

This domino-type game promotes the memorization and linkage of the addition/subtraction facts, the use of mental math, and trial-and-error thinking to balance addition and subtraction number sentences. *The game may be played when students are mastering the subtraction facts or at any point during the unit.*

As an alternative to having students actually play the game, the game cards may be used in an activity setting. Have students work in small groups. Have each group display all 32 game cards face up. As a group, have them match all 32 number sentences with their respective solutions. Because each solution (on a card) matches exactly one number sentence (on a card), and because there is exactly one solution (on a card) for each number sentence (on a card), this activity is akin to completing a puzzle.

Instructional Game: Balance the +/– Number Sentence!

Rules

1. **Getting Started:** Students work in groups of two or three. Each group is given one copy of the 32 game cards on pages 156 through 157. Ask students to cut out the game cards and thoroughly shuffle them. *Have them deal 5 cards to each player.* The remaining cards are placed facedown in a stack in front of them. *Allow time for the players to study the cards in their hand.* Explain that each card contains a number sentence on the left and a solution to a different number sentence on the right (shown by the white number).

2. **Playing the Cards:** The first player (Player A) begins by placing one card from his or her hand face up on the table. Player B must try to match either end of that card as follows.
   - Player B may place a card to the left of Player A's card so that the white number on Player B's card provides the value of □ on Player A's card.

     Or

   - Player B may place a card to the right of Player A's card so that the number sentence on Player A's card is solved by the white number on Player B's card.

3. **Sample Rounds:** Suppose there are two players and they have the cards shown below. If Player A plays the card on the far left, Player B should play the card on the far right because 14 solves □ – 8 = 6. Then, Player A should play the second card from the left because 15 – □ = 8 is a number sentence for which 7 is the solution.

□ – 8 = 6	7	15 – □ = 8	6

Cards in Player A's Hand

□ – 9 = 2	6	17 – □ = 9	14

Cards in Player B's Hand

The correct placement of the cards on the table for the sample rounds is shown below. Note that when a card is placed on the table, the matching sides of the cards should touch.

17 – □ = 9	14	□ – 8 = 6	7	15 – □ = 8	6

Balance the Number Sentence! (page 2)

4. When You Cannot Match a Card on the Table: If a player is unable to match either end of any card on the table, he or she must draw a new card from the stack. During that turn, the player should then try to use the drawn card to match either end of any card on the table. If that player is unable to play the card, the player must then place any card from his or her hand on the table to begin a new row.

5. **Placement of an Incorrect Card:** If a player places an incorrect card on the table, the card must be returned to the player's hand, and play resumes with the next player.

6. **Scoring Points:** A player earns one point for each card he or she *correctly* places on the table to *solve a problem.* No points are scored if a card is placed incorrectly or if a player has no cards that match and must place a card on the table to begin a new row. Note: During a round, a player may place a card to solve a problem in any of the rows.

7. **When You Have No Cards in Your Hand:** If a player begins a round with no cards in his or her hand, the player draws a card from the stack and attempts to play it. If he or she is unable to play it, the player begins a new row as per Rule 4. If no cards are left in the stack for the player to take, the player loses that turn.

8. **How the Game Ends:** The game ends when no player is able to play a card and there are no more cards left in the stack. The player with the most points wins.

Note that some sentences on the game cards contain more than one □. Advise students that each □ in a given sentence stands for the *same number.*

Game Cards for Balance the +/– Number Sentence!

When more than one □ appears in a sentence, each □ stands for the same number.

□ – 8 = 6	7	17 – □ = 9	14
15 – 9 = □	8	□ – 9 = 2	6
5 + □ = 14	11	14 = 9 + □	9
21 = □ + 0	5	19 – □ = 0	21
23 + □ = 23	19	19 = 22 – □	0
□ + 19 = 23	30	100 = □ + 40	4
□ + □ = 14	60	2 + □ + □ = 22	25
24 + 25 = □ + 24	100	□ – 15 = 15	3

Game Cards for Balance the Number Sentence! (page 2)

$199 - \square = 100$	10
$13 + \square = 12 + 13$	16
$\square - \square = 13 - \square$	18
$49 + 49 = 100 - \square$	15
$9 + 8 + 7 = \square + 7$	23
$49 = \square + 99 - 99$	2
$31 - 24 = 31 - \square$	22
$18 + \square = 40$	1

$\square - 50 = 50$	20
$\square - 4 = 5 + 6$	12
$23 + \square = 47 - 1$	13
$16 + 0 = 0 + \square$	99
$36 - \square - \square = 0$	17
$60 = \square + \square + \square$	49
$\square - 22 = 18$	24
$51 - 9 = 50 - 9 + \square$	40