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Circling Circles

The Lesson (Grade 3)

A close examination of algebra's origins reveals two roots—*arithmetic* and *geometry*, hence the powerful interrelationship between arithmetic, geometry, and algebra. In recent work by advocates of teaching algebra in the early grades, much has been written about the connections between arithmetic and algebra (Carpenter, Franke, & Levi, 2003; Carraher, Brizuela, & Schliemann, 2007; Russell, Schifter, & Bastable, 2011). My contention is that not enough has been written about the privileged relationship between *geometry* and algebra. Geometric concepts are as vital as numeric ones, and their links with algebra are tangible in ways that numerical concepts are sometimes not. Children in grades 3–5 are still in the concrete operational stage, according to Piaget's theory of cognitive development. So the concrete experiences of working with shapes—manipulating, tracing, drawing, measuring, discovering properties, examining characteristics, and noticing patterns—are important components of children's mathematics education.

Experiences with geometry develop students' spatial sense. This includes spatial intuition and spatial perception, two potent tools according to Fields Medal recipient Sir Michael Atiyah. They are the reason

> geometry is actually such a powerful part of mathematics—not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition: our intuition is our most powerful tool. (Atiyah, 2001, p. 5)

Children come to school with intuitions about, and insights into, 2-D and 3-D shapes, and with ideas about their properties and interrelationships. Coupled with their intuitive knowledge is children's natural interest in things geometric: They are attracted to and intrigued by squares, circles, and triangles, fascinating and familiar shapes in their geometric world. The *Circling Circles* Exploration offers opportunities for cultivating algebraic reasoning through the exploration of the most perfect, plane, closed curve called the circle, considered by ancient and modern cultures the symbol of the human psyche, the unity of heaven and earth, the inclusivity of the universe, and much more.

The two-day lesson recounted below took place in a third grade classroom of an elementary school I had been working with for three years. Our common goals were to deepen teachers' content and pedagogical knowledge of mathematics and to create a culture in which teachers and students valued, discussed, and enjoyed meaningful mathematics, both in and out of the classroom.

REFLECT . . .

> [Friedrich Froebel's] philosophy was clear: if children could be stimulated to observe geometric objects from the earliest stage of their education, these ideas would come back to them again and again during the course of their schooling, deepening with each new level of sophistication. The rudimentary appreciation of shapes and forms . . . would become more refined as students developed new skills in arithmetic and measurement and later in more formal algebra and geometry.
>
> —Thomas F. Banchoff (1990, p. 11)

Setup

On a winter morning, the lesson was intended to model Common Core mathematical practices (NGA/CCSSO, 2010). I chose *Circling Circles* because it incorporates several: Modeling with mathematics, using appropriate tools strategically, looking for and making use of structure, and looking for and expressing regularity. Regarding content, while the lesson clearly is about geometry, it also features algebraic concepts as well as algebraic ways of reasoning, talking, and doing. Just as in the field of mathematics, where algebra is enhanced by geometry and geometry is expressed through algebra, this lesson interweaves both ways of knowing.

The third-graders were quietly waiting. Seated in groups of two or three students, each pair or trio had a bag with all the necessary manipulatives (Figure 1.1).

Ms. Flores, a caring and dedicated teacher, welcomed me back to her classroom. The Hula-Hoop I had requested was hidden under her desk. Inspired by my theatre experience, I enjoy adding elements of surprise or suspense to my teaching, whenever possible. It was 10 o'clock, and I was granted a full hour.

FIGURE 1.1 Manipulatives needed for the lesson

Photo by Didier Rousselet

Discussion

1. Exploring Circles

In my lesson opening, I wanted to ascertain students' ideas about circles: "What can you tell me about circles?" "It's like a ball," began Daniel. Following suit, Daniel's peers soon had a list of circular objects cascading onto the white board: bicycle wheels and car tires, CDs and DVDs, soccer balls and basketballs, jar lids, checker pieces, plates, a roll of tape, coins, and more (Figure 1.2). We listed these contributions under the heading "Circular (Round) Objects," highlighting that *circus* and *circle* have the common root *circ* from the fact that original circuses had circular arenas for human and animal performances.

FIGURE 1.2 Examples of circular (round) objects

Photo by Didier Rousselet

Dimension

I drew students' attention to the different dimensions of the listed objects. They knew the term *3-D* from movies, so we connected the letter *d* to the word *dimension.* They agreed that CDs and coins are *flat* and could live in *Flatland*. Since they hadn't heard of that word, I held up the famous little book of the same title by English schoolmaster Edwin Abbott (1992). "In this Romance of Many Dimensions (the book's subtitle), the inhabitants of Flatland are line segments and polygons," I explained. Acknowledging her students' apparent interest, Ms. Flores promised to order the book so they could read it later. We returned then to our circles.

Soccer balls and basketballs on the other hand are definitely 3-D objects. They cannot live in Flatland, because they have "thickness." They live in *Spaceland*, also coined by Abbott. We reviewed the word *sphere* for a 3-D, hollow, inflated ball like the classroom globe or a beach ball. And we learned the word *disk* for a 2-D or flat, circular object, almost like a CD or DVD except with neither thickness nor a hole at the center.

FIGURE 1.3 Drawing a circle with marker and string

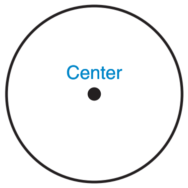
Photo by Didier Rousselet

Definition

Just as the 2-D Square was the main character in the book *Flatland*, the 2-D Circle was the protagonist of our lesson. "I can see you know lots of circular objects. Now, can someone explain what a *circle* is?" As they pondered how to put their thoughts into words, I pulled out of my bag of props a prepared piece of string with a black marker attached to one end: my handmade compass. Then I drew a point on the board, labeled it CENTER, and holding the free end of the string at *that* point with my left thumb, I drew a circle with the marker in my right hand (Figure 1.3). Another perplexity arose: Was the circle the black ring only, or the ring *plus* the enclosed circular region of the white board?

To resolve the quandary, we accessed the online dictionary, Math is Fun, on the interactive white board and looked up the definition of the word *circle:* "A 2-dimensional shape made by drawing a curve that is always the same distance from a center" (www.mathsisfun.com/definitions/index.html).

FIGURE 1.4 A circle and its center



The class accepted that the circle was therefore "the black ink only," as one student put it, *not* the enclosed circular disk, nor the point called *center of the circle* (Figure 1.4). "Is that the same for triangles?" inquired bright-eyed Disha. "Excellent question! You took the words right out of my mouth!" I exclaimed. Disha was a math lover and a high-level thinker. Drawing a triangle on the board and shading it in, I explained: "Even though we often refer to *all of this* as a triangle, *mathematically* a triangle is only the three connecting line segments, not the shaded region inside." That was news to some students.

RECOGNIZE . . .

A geometric object is a mental object that, when constructed, carries with it traces of the tool or tools by which it was constructed.

—Nathalie Sinclair, David Pimm, & Melanie Skelin (2012, p. 8)

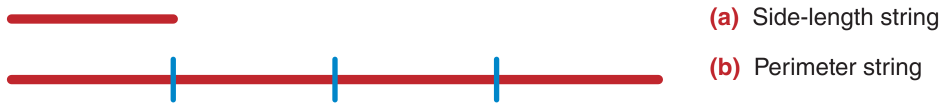
2. Posing the Problem

"I brought a problem for you today about circles; are you ready to solve it?" They were. "Before I state the problem, I have a warm-up question."

Modeling With a Square

Ms. Flores assisted me by holding up another prop: a square picture frame. I took out a precut piece of string, equal in length to the side length of the square frame. Modeling a side length, I placed it atop the frame, stretching it from corner to adjacent corner, and asked, "How many times do you think this piece of string fits *around* the square frame?" Several hands shot up. I called on Paula: "Four times," she answered. "Who agrees?" I continued. More hands went up. "Who can convince us all that it's four times?" I carried on. "It's four 'cause they're four equal sides," explained Emma. It sounded convincing. I nevertheless modeled the *perimeter*, as I had the side length, by wrapping a longer string tightly *around* the frame and cutting off a piece equal in length to the frame's perimeter. We compared lengths: Indeed, the short string fit four times "inside" the long one. I marked the perimeter string to model that it contained four side-length strings (Figure 1.5).

FIGURE 1.5 Materializing the square frame's side length and perimeter with string



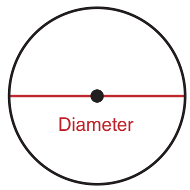
Modeling With a Circle

It was time to bring out the Hula-Hoop. The kids got excited. Some of them wanted to show me how well they could Hula-Hoop. I promised they could do so *after* the lesson. Extra incentive to dive into our problem.

Diameter. We pretended the hoop was as thin as the black circle drawn on the white board. The students understood that real-world circles must have thickness; otherwise they would be too fragile to hold. Manipulatives are never perfect, but it suffices to point out their limitations and then proceed. Children are experts at pretending. Plus, they love to indulge teachers. "Any idea what the *diameter* of a circle is?" I began. Some knew the word but couldn't define it. "Breaking down the word, *dia* means "going through or across" in Greek, and *meter* means "measure." So the diameter is the *measure* of *what*, going *through what*?" I asked, as I turned to the circle on the board to hint at the center point. Some caught on. "The *center?"* asked Alexander? "Correct. But *what* goes through the center?" I pressed. "A piece of string?" added Alexander. "Yes, a taut piece of string we'll call a *line segment*," I explained further. "So the diameter is a line segment through the center of a circle that touches the circle at both ends," I clarified.

As Ms. Flores held the Hula-Hoop firmly, I stretched a piece of string across the hoop, through its invisible center, and touched the hoop at both ends with my index fingers and thumbs. We estimated the center's location, aided by a visual clue: The diameter had to divide the circle into *two equal semicircles* (Figure 1.6). Students helped me with their view of the hoop from a distance. I finally cut off a piece and held it up saying, "This is a representation of the Hula-Hoop's diameter. We'll call it our 'diameter string.'"

FIGURE 1.6 A circle and its diameter



The Question

"And now for my question: How many diameter strings would we need to encircle the Hula-Hoop? In other words, how many times do you think the diameter fits along the perimeter of this circle?" Before I could explain that the perimeter of a circle has a special name, Kindra corrected me, "You mean the *circumference*?" "Absolutely," I intoned with praise, "the distance *around* a circle is called the *circumference*—another word starting with *circ,* right?" No doubt Cindy Neuschwander's math adventures of Sir Cumference were popular in Ms. Flores's class (Neuschwander, 1997).

Student conjectures ranged from 2 times to 5 times. Ms. Flores pulled up the prepared chart on the interactive board and tallied their conjectures by counting raised hands for each category (Table 1.1). There were more 3s and 4s than there were 2s and 5s.

TABLE 1.1 First chart used to tally student conjectures

How many diameter strings fit around the Hula-Hoop?			
2 times	**3 times**	**4 times**	**5 times**

Not surprisingly, only whole numbers were uttered. After all, the students were only beginning to learn that numbers called *fractions* live between numbers 0 and 1 on the number line, and between other consecutive whole numbers as well. Moreover, the answer to the warm-up question on the perimeter of a square was the whole number 4.

Does the Size of the Circle Matter?

Before I sent them off in small groups to explore the problem, I surveyed one more thing: "Do you think the number of times the diameter fits around a circle will change if the circle gets bigger or smaller?" All but one student opined that the answer

TABLE 1.2 Second chart used to tally student conjectures

What do you think?	
I think the answer is the same for all circles	I think the circle size matters: larger circles give greater answers.

depends on the circle size, namely, that in larger circles the answers would be greater than in smaller circles. We tallied those responses as well, admiring Jason's daring to be the sole tally mark in the first column (Table 1.2).

3. Measuring to Find Out

Measuring With String

Students removed scissors, string, marker, and pencil from their bags, as instructed. Each group received a different-size circular object and the same instructions (Figure 1.7; see Appendix, page 205, for student worksheet). The eight circular objects handed out were lids ranging in size from 3 to 29 cm in diameter.

FIGURE 1.7 Circling circles instruction sheet (Part I)

Circling Circles (Part I)

Names: __________ __________ __________ __________

Prediction: Before measuring, discuss the number of times *you think* the diameter will fit around the lid. Agree on a number and write your group's prediction here __________.

Tasks: Decide who will do Steps 1, 2, 3, and 4.

Step 1: Cut a piece of string equal in length to the diameter of the lid.

Step 2: Cut a piece of string equal in length to the circumference of the lid.

Step 3: Use the marker to mark the number of times the short piece of string fits inside the long piece of string.

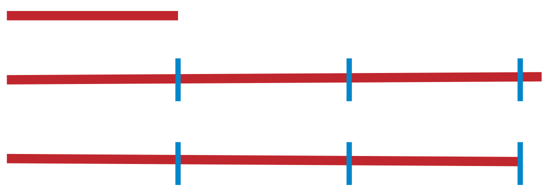
Step 4: Talk to each other and share your thoughts. Choose one idea and write about it on the back of this paper. It can be a question, an observation, a conjecture, a picture, or something else.

We read the directions together. "Thumbs up if they are clear; thumbs down if you need help understanding something." Ms. Flores and I attended to those in need while the others got to work. One group worked on Ms. Flores's wastepaper basket (35-cm diameter) and another on my child-size Hula-Hoop (71-cm diameter). Students assigned to the basket were instructed to measure the container's rim, at the opening. That required dexterity, as did measuring the contour of the Hula-Hoop!

Almost all groups found that the diameter fit "three times and a little more" around their circular objects. The majority marked their strings as shown in Figure 1.8 (b), and a few as shown in Figure 1.8 (c).

That the diameter length would fit inside the circumference length *the same number of times for all circles* was "awesome" for some and "cool" for others. One child—a little man in a boy's body—who loved big words called out, "astonishing!" The discovery of the invariant ratio between circumference and diameter in *all* circles was a true "aha moment" for the class, as almost all students had conjectured higher ratios for larger circles.

FIGURE 1.8 **Materializing the lid's diameter and circumference with string. The intervals between consecutive tick marks in (b) and (c) represent the lengths of the diameter string.**



Ratio. We didn't use the word *ratio.* Instead I chose the word *relationship.* While the circle sizes varied, we found that the *relationship* between the length of the diameter and the length of the circumference remained the same. Ms. Flores's students had discovered an important property of circles in the same way that early civilizations had, namely, by noticing that a rope around the periphery of a circle equaled just over three lengths through its center. That discovery marked the beginning of the four-million-year story of pi; it also marked the end of our *Circling Circles* Exploration, Part I. My third-graders were tired but looking forward to further investigation on Day 2.

REMEMBER . . .

From the very beginning of his education, the child should experience the joy of discovery.

—Alfred North Whitehead (1916, p. 1)

Measuring With a Metric Tape

Circling Circles Part II, which took place the following day, began with a hands-on exploration. Students used their metric measuring tapes and followed a second set of instructions (Figure 1.9; see Appendix, page 205, for student worksheet). Again we read the instructions aloud and provided clarification. We reviewed a centimeter–body connection: "A centimeter is about as wide as a third-grader's pinky finger (at the top), and an inch is about two pinky fingers."

Uncertain about the students' grasp of the millimeter tick marks on the metric tapes, I told them to record the whole-number centimeter measurement nearest to the length they measured. However, if the measurement fell halfway between two consecutive centimeter measures, they were to record the *larger of the two* numbers. (In other words, they would round up). Ms. Flores and I roamed the room, assisting students who called for help. When all groups were finished, we recorded their data in a class table in the order in which they raised their hands (Table 1.3). Students accepted the use of *D* and *C* quite naturally.

FIGURE 1.9 Circling circles instruction sheet (Part II)

Circling Circles (Part II)

Names: __________ __________ __________ __________

Tasks: Decide among yourselves who will *estimate* (Steps 1 and 3), *measure* (Steps 2 and 4), and *write* (Step 5).

Step 1: *Estimate* the length of the diameter (the longest distance **across**): ___________ cm

Step 2: Now use the metric tape to *measure* the length of the diameter: ___________ cm

Step 3: *Estimate* the length of the circumference (the distance **around**): ___________ cm

Step 4: Now use the metric tape to *measure* the length of the circumference: ___________ cm

Step 5: Talk to each other and share your thoughts. Choose one observation or reaction and *write* it on the back of this paper.

Delving Deeper

4. Representing With Numbers and Words

Saying It With Numbers

Once the measurements from all groups were displayed, I invited observations on the numerical data. More specifically, I asked the students to focus on how the numbers in the right column compared to the numbers in the left column. As with any group of students, comments ranged from lower-level, specific remarks such as, "Group H's numbers are backwards [referring to 29 and 92]," to higher-level, more general observations such as Paula's comment: "If you take the first number and multiply by three, you sometimes get the second number," noticing the pairs (5, 15) and (6, 18). Asked to give examples, she obliged me, "Five times three is fifteen, and six times three is eighteen." "Any others?" I inquired. "No," she replied. "What about the other pairs of numbers," I asked Travon, confident I could forge ahead with his help, having overheard his conversation with his partner. Eloquently he proceeded, "Well, if you multiply by three, you *almost* get the second number," with a nice long drawl punctuating "almost." "What do you mean by 'almost'?" I continued. "You get a little bigger . . . like, um, three times three is nine but it's ten. Four times three is really twelve, but it's thirteen, and, um . . . thirteen is a little bigger than twelve." I recorded Travon's reasoning (Figure 1.10).

TABLE 1.3 Class data table

Group	Diameter *Length across* (in cm) *D*	Circumference *Length around* (in cm) *C*
A	3	10
B	4	13
C	7	22
D	11	35
E	6	18
F	8	26
G	5	15
H	29	92
I (Hula-Hoop)	71	227
J (wastebasket)	35	112

Observing Regularity in *All* Rows. Not convinced that all students could work out mentally the relationship between the double-digit numbers, I called on Robin for further clarification: "Can you show the class that the same relationship holds between the two measurements found by Groups D and H?" Robin rose to the call: "Eleven times three is thirty-three—that's easy—and thirty-five is a little bigger," she explained, borrowing the phrase "a little bigger," used by Travon and recorded on the board. Displaying her amazing number sense, Robin continued: "Well, twenty-nine is close to thirty, and ninety-two is close to ninety, and three times three is nine—oh, no, I mean, thirty times three is ninety." Again, for the purpose of clarity, I recorded Robin's thinking right under Travon's, hoping the regularity in the data would emerge for *all* students (Figure 1.11).

FIGURE 1.10 Travon observes that circumference measurements are "a little bigger" than three times diameter measurements.

D = 3	*C* = 10	3 × **3** = 9 / 10 is a little bigger than 9
D = 4	*C* = 13	4 × **3** = 12 / 13 is a little bigger than12

FIGURE 1.11 Robin observes that the relationship between circumference and diameter is approximately the same for several pairs of two-digit measurements.

	D = 3	*C* = 10	3 × **3** = 9 / 10 is a little bigger than 9
	D = 4	*C* = 13	4 × **3** = 12 / 13 is a little bigger than12
	D = 11	*C* = 35	11 × **3** = 33 / 35 is a little bigger than 33
	D = 29	*C* = 92	
	↓	↓	
Estimating:	Close to 30	Close to 90	30 × **3** = 90 / 92 is a little bigger than 90

Together we examined the data in the last two rows of the table and confirmed the *almost*-tripling relationship between *D* and *C* in *all* rows.

By now, everyone could see the relationship between the numerical measurements of diameter and circumference from the class data table, but clearly *not everyone* had made the connection between this numerical relationship and the string experiment from Day 1. This was evidenced by the Eureka wave that spread across the room following Lynnette's exclamation, "Oh, neat, that's just like the string!" Eager to have everyone share in the joy of this connection, I purposely pursued, perplexed, "What do you mean?" "The long string was almost like three times the short string, but a little more!" Which just goes to show us teachers that sometimes what we *think* is implicit needs to be made explicit. With their sometimes-unexpected comments and often-amazing questions, students give us all the ingredients to make a good lesson . . . provided we remain attentive listeners!

Saying It With Words

I explained to the class why Groups E and G got a relationship of *exactly* three to one. In a perfect world, they too would have obtained *a little more than* three. Students need to know that any data set contains measurement errors, and that, in particular, our investigation was subject to human error in measuring, reading, and recording; imperfection in measuring instruments; effects of rounding numbers up or down; thickness of the Hula-Hoop and basket; and other factors. But, these errors were normal and by no means minimized the importance of their discovery.

On Day 1, through the concrete string experiment, students touched upon the discovery through visualization. On Day 2, producing classroom data through measurement revealed the same discovery, but his time through the language of numbers. Now, their next task was to translate their discovery into the language of words—into English. Instructions were clear: Their sentences had to include the words *diameter* and *circumference* and had to describe a property of *all* circles. To add incentive, I shared the fact that the first recorded work in mathematics—the Rhind Papyrus, currently in the British museum—contained mathematicians' fascination with this same property of circles. And that was about 1650 BCE, more than 3,500 years ago, in Egypt!

After a few minutes, students volunteered to read their generalizations out loud. Ms. Flores selected five of them for their originality and copied them verbatim from slate to board (Table 1.4). Although Sentence 2 didn't comply with the directions, it helped visualize the string experiment from Day 1. These verbal representations reinforced the observed relationship between the diameter and circumference of all circles: big, small, *all*. They also inspired the less verbal students to carve out their own formulations.

TABLE 1.4 Verbal representations of the diameter-to-circumference relationship

Examples of Verbal Representations
1. When you take a circle and mezur the diameter you get the circomfrence times 3 but thers a bit more.
2. If you walk round a circle it's like a little more than 3 times if you walk through the centr.
3. The diametr string of a circel needs three to go round the circumference.
4. 3 times the diameter plus a wee bit more is the sircumference
5. diameter + diameter + diameter = circumference almost

Multiplication–Division Connection

I projected my next slide onto the board, adding a new, empty column to the right of our class data table. It was titled "$C \div D$ or $\frac{C}{D}$." Using their TI-108 calculators, the students divided C by D. My goals for them were to have fun dividing with calculators, to expose them to decimals in a novel way, and to observe the inverse relationship between multiplication and division.

The excitement over using calculators (to divide, no less!) was coupled with the puzzlement caused by the digit strings of variable lengths. The ten group recorders called out the numbers appearing on their calculators. Ms. Flores entered them into the last column of the table (Table 1.5), diligently transcribing all digits called out. Two groups had inadvertently reversed dividend and divisor, but they quickly caught on.

Curiosity engendered comments such as, "They all start with a three," and "Some are short and some are long!" and questions such as "Are the longer ones bigger?" This wonderful question from Lynnette brought a smile to my face; listening to children's math talk invites us inside their minds and reminds us how they think. The question prompted me to draw a magnified interval of the tape measure on the board and show that all the quotients found, while in *appearance* very different, live close to one another on the number line (Figure 1.12 red segment). It was new to many students that the small tick marks between

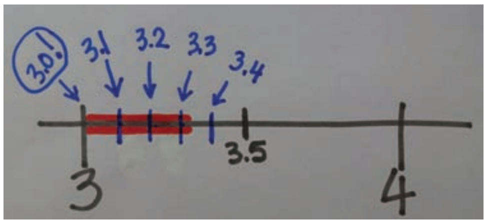
TABLE 1.5 Students use the division key on four-function calculators to find $C \div D$

Group	Diameter/*D* *Length across* (in cm)	Circumference/*C* *Length around* (in cm)	$C \div D$ or $\frac{C}{D}$
A	3	10	3.3333333
B	4	13	3.25
C	7	22	3.1428571
D	11	35	3.1818181
E	6	18	3
F	8	26	3.25
G	5	15	3
H	29	92	3.1724137
I (Hula-Hoop)	71	227	3.197183
J (wastebasket)	35	112	3.2

three and three-and-a-half were labeled 3.1, 3.2, 3.3, and 3.4. But it made sense. Also, they easily accepted "3.0" as an alternative name for "3," as it logically fit into the sequence, 3.0, 3.1, 3.2, 3.3, 3.4, 3.5 . . . Third-graders have plenty of time to deepen their understanding of decimals in grades 4 and 5, but visualizing them on the number line from early on is crucial.

Having established "a little more than three" first with the string experiment and second in the multiplicative relationship between *D* and *C* (Table 1.3), some students were not surprised when encountering it again, especially those who understood that division is the inverse of multiplication—"like subtraction and addition," said Madison. But, while this is important, I chose to move on to demystifying this mysterious number.

FIGURE 1.12 All quotients in Table 1.5 live within the small red interval on the number line.

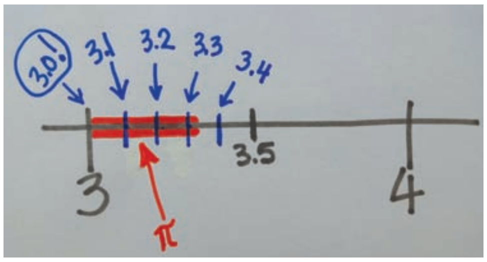


5. Representing With Pictures and Symbols

Pi Talk

Hoping the students were clear on the existence of human error in measurement, as discussed, I posed a question—albeit a low-level, yes-no question—to the entire class to double check: "How many of you think that in a perfect world we should have all found the same number?" The majority of hands sprang up. "Any idea what that number would be, Lamar?" I asked pointedly, secretly knowing from Ms. Flores of his fascination for pi. "A little more than three?" he replied. "Absolutely. But do you know a special number that begins with three and has many digits beyond the decimal point?" I persisted. "Pi?" he asked excitedly, clearly never having connected pi to circles. I pressed a little further,

FIGURE 1.13 Locating pi inside the interval of quotients computed by the students



"What number is pi?" "Three point one, four, one, five, nine, two, six, five, three . . . " he answered, impressing students and teachers. "Excellent!" cried Ms. Flores, so proud of her student." I carried on in admiration, "Wow! That's indeed the number you *all* would have found in a perfect world." Unable to contain his enthusiasm, Lamar blurted out, interrupting me, "Pi has a million numbers. My Dad taught me." Feelings of wonderment could be seen, felt, and heard.

To wrap up our "pi talk," we pulled up the "100,000 Digits of Pi" website (www.geom.uiuc.edu/~huberty/math5337/groupe/digits.html) and scrolled down. The students were mesmerized. The hardest concept to grasp was that *more and more* digits did not signify a *bigger and bigger* number. "Is pi way bigger than all the quotients we found?" I inquired. "No!" said Kayla, assertively. "Do you agree with Kayla?" I asked Gabe, who was daydreaming. "I don't know," he replied, pretending to be reflecting. "Then call on someone to help you understand why this number, with so many digits, is *not* a huge number," I persevered. Gabe called on our Number Devil. Once again, Robin met the challenge with confidence, "Because it starts with three and then there's a one, so it's smaller than three two five (meaning 'three *point* two five')!" "Good thinking! No matter how many digits there are—even a million—pi lives between 3.1 and 3.2, like the four other numbers we found with our calculators," I said, pointing to numbers such as 3.1818181 in our table. "So 3.25 and 3.3333333 are *bigger* than pi!" I reiterated, indicating pi's relative location inside our interval with an arrow (Figure 1.13).

I didn't worry whether this concept was clear; over the following two years, the students would grow deeper roots of understanding of decimals.

REALIZE . . .

One can get a tremendous amount of mileage out of a continuing discussion on the estimation of π, from the first time a kindergarten student realizes that the belt around a can reaches a little more than three times across the top, to second-semester calculus where one studies integrals for arc length.

—Thomas F. Banchoff (1990, p. 35)

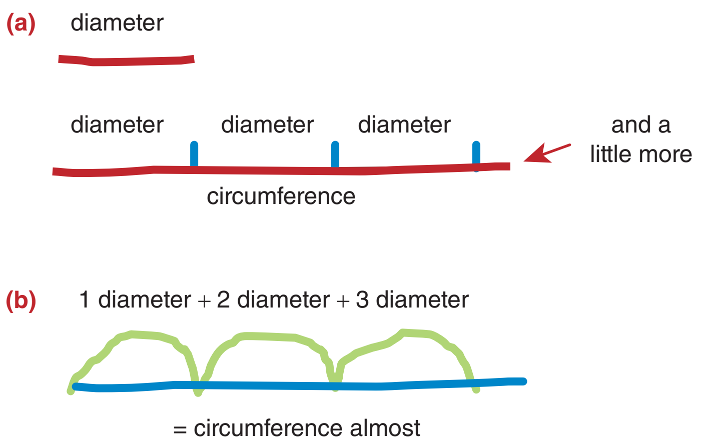
Saying It With Pictures

For the final task, students had a choice: Those who wanted to engage their creative *artistic* talents joined Group 1. Their task? To "say it with pictures"; namely, to draw a picture representing what they learned about diameter and circumference in all circles. Ms. Flores provided these students with drawing supplies and assistance. Group 2, a smaller group of math aficionados, opted to use a different creative talent: *abstract thinking*. Their task? To "say it with symbols"; namely, to write an equation relating C, D, and π. (Students learned the symbol for pi from the website.) They worked with me.

Ms. Flores handed out colored paper, colored pencils, and crayons. She also allowed students to use slates and colored dry-erase pens if they preferred. She suggested tracing a lid's contour if their representation needed a circle. Once they understood the assignment, students were on their own, guided solely by their imagination and creativity.

Student-Created Pictorial Representations. Students were clearly inspired by the prior day's actions of cutting diameter and circumference strings, comparing them side by side like line segments, and marking the diameter-string units inside the circumference strings. Consequently, most pictorial representations depicted these actions. Max and Danica's pictures, faithfully reproduced here, are representative of two variations on the "linear string theme" observed (Figure 1.14).

FIGURE 1.14 Reproductions of sample student drawings of the *D*-to-*C* relationship. (a) Max's drawing includes both diameter and circumference lengths and their relationship; (b) Danica's drawing shows the circumference string as composed of three diameter strings and gives a quasisymbolic equation.

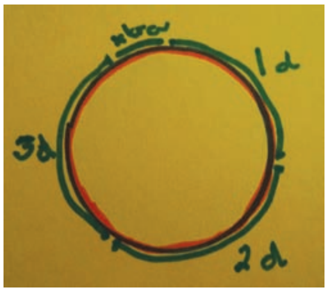


Only one student, Gabe, depicted the visual "wrapping" of the diameters *around* the circumference of a circle, which had been verbalized and modeled by the teacher but not enacted by the students. Figure 1.15 is a reproduction of his clever rendering. As the saying goes, this picture is worth a thousand words. Gabe's drawing not only enlightens the viewer but also reveals the significant learning that took place in two days. Recall that only one day earlier, the children had guessed that the diameter would wrap around a circle two, three, four, and five times. Gabe's rendition was a reminder for me not rush to conclusions about daydreamers. After all, 17th century French mathematician René Descartes claimed that analytic geometry flashed before him in a dream, and 20th century Indian mathematician Srinivasa Ramanujan that new mathematical ideas came to him during meditation!

Saying It With Symbols

After two days of using *C* and *D* as shorthand notations for circumference and diameter, students were comfortable with their meanings. "It's like our initials instead of our

FIGURE 1.15 Author's reproduction of Gabe's pictorial representation that vividly conveys the fact that it takes three diameters—plus a little "xtra"—to wrap around the circumference.



names," offered Disha as a lovely analogy during our small-group discussion about symbols. Less obvious to these second-semester third-graders was the meaning and use of the recently acquired symbol, π. "It's actually a number close to 3—between 3.1 and 3.2 to be exact—with millions of digits beyond the decimal point; since it's impossible to say or write the entire number, we use π for short." That was my attempt at explaining pi to young children. Some students seemed comfortable with this abstraction; others less so. Writing number sentences was common in Ms. Flores's class, but writing symbol sentences was less so.

Teacher-Guided Symbolic Representations. To guide them on the final stretch, I returned to the verbal representations on the board (Table 1.4) and had them pick the ones they found easiest to translate into equations. "3 times the diameter plus a wee bit more is the sircumference" and "Diameter + diameter + diameter = circumference almost" were their choices. Both were partially in equation form, as the first contained "is" and the second an equals sign. For lack of time, I focused on the first sentence, which they seamlessly converted to, "$3 \times D$ + aweebitmore = C." "So three times D doesn't quite equal C, right?" I said. They acquiesced. Nudging them further to the finish line, I inquired, "So what could we multiply D by to get C *exactly*?" "Pi?" guessed Disha, with a radiant smile. I felt it was more of a guess than an understanding, but, after all, she was only in third grade! We accepted her proposition, and she proudly wrote our final equation on the board (Figure 1.16). I added the equation $3^+ \times D = C$ next to it, adding, "'Three plus' will be our secret code to remember that pi is *a little more than* three, OK?" They smiled in agreement. In time, both equations will merge and π will take on meaning. But for now, the *about-3 relationship* between D and C was what mattered.

FIGURE 1.16 Symbolic representations of the *D*-to-*C* relationship

$\pi \times D = C \quad 3^+ \times D = C$

Closure

Ms. Flores and I brought closure to the *Circling Circles* Exploration by having the two groups visit each other's work and learn from their peers' representations. In the final whole-group discussion, students took turns verbalizing what they had learned. The actions of circling, stretching, cutting, comparing, measuring, computing, tracing, drawing, and writing were now stored in their bodies' memory banks. In addition, students had cultivated habits of mind essential to mathematics, including conjecturing, discerning, comparing, explaining, discovering, verbalizing, representing, generalizing, and even symbolizing. By acting, thinking, and talking like young mathematicians, they had planted the seeds of a fundamental concept that in a couple of years would lead to the algebraic notions of ratios, proportional relationships, and linear functions. The mysterious number pi—and its relationship to circles—had begun to be demystified.

Sandra had not forgotten: "Can we do the Hula-Hoop now?" "Certainly!"